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LINEAR ALGEBRA PRACTICE

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Посібник

«LINEAR ALGEBRA PRACTICE»

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FOREWORD

The section "Linear Algebra" is a part of the course "Higher Mathematics", which is studied by students of technical specialties of the University.

The «Linear Algebra Practice» provides theoretical information that covers topics such as Matrices, Matrix Operations, Determinants, Inverse Matrix, Matrix Rank, Systems of Linear Equations, as well as tasks for independent work.

This publication is aimed at helping English-speaking students to consolidate their knowledge and skills in solving linear algebra problems.

1. ELEMENTS OF LINEAR ALGEBRA

1.1. Matrices. Matrix Operations.

A rectangular arrangement of numbers a_{ij} ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) into m rows and n columns written by

$$A = A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

is called a **matrix**.

The **dimensions** of a matrix tells its size: the number of rows (m) and columns (n) of the matrix, *in that order*.

Two matrices have the same size, if their dimensions are equal.

The individual items (numbers, symbols or expressions) in a **matrix** are called its **matrix elements** or **entries**. The entry in the i -th row and the j -th column of a matrix A is denoted by a_{ij} . The subscripts indicate the row first and the column second.

A short form the matrix is indicated $A = (a_{ij})$.

A matrix with one row is called a **row matrix**: $(a_{11} \ a_{12} \ \dots \ a_{1n})$.

A matrix with one column is called a **column matrix**: $\begin{pmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{n1} \end{pmatrix}$.

A **square matrix** has as many rows as columns, the number of which determines the order of the matrix, that is, an $n \times n$ matrix is the matrix of the n -th order.

Two matrices, $A = (a_{ij})$ and $B = (b_{ij})$, are **equal**, if they have the same sizes and their elements are equal by pairs: $a_{ij} = b_{ij}$ ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$).

In a square matrix $A = (a_{ij})$, the elements a_{ii} , with $i = 1, 2, 3, \dots, n$, are called the **diagonal elements of the matrix A**. The **main diagonal** (sometimes **principal diagonal**, **primary diagonal**) is the list of entries a_{ii} , $i = \overline{1, n}$.

A square matrix is called a **diagonal matrix**, if off-diagonal elements are equal to zero or, symbolically, $a_{ij} = 0$ for all $i \neq j$.

A matrix is called a **zero-matrix** (0-matrix), if it consists of only zero elements: $a_{ij} = 0$ for each (i, j) . In a short form, a zero-matrix is written as 0.

A diagonal matrix whose diagonal elements are equal to unity is called an **identity matrix**. It is denoted by E .

A square matrix has a **triangular form**, if all its elements below the leading diagonal are zeros: $a_{ij} = 0$ for all $i > j$.

If $A_{m \times n} = (a_{ij})$ and $B_{m \times n} = (b_{ij})$ are matrices of the same size, then the **sum**, $C = A + B$, is the matrix

$$C_{m \times n} = (c_{ij}) = (a_{ij} + b_{ij}).$$

Any matrix A can be multiplied on the right or left by a scalar quantity k . The **product** is the matrix B (of the same size as A) such that

$$B = kA = Ak = (k \cdot a_{ij}).$$

If A and B are matrices of the same size, then the **difference**, $A - B$, is the matrix

$$A - B = A + (-1)B.$$

The **product of two matrices**, A and B , is defined, if and only if the number of elements in a row of A equals the number of ones in a column of B . Let A be an $m \times k$ matrix and B be an $k \times n$ matrix. Then the product AB is the $m \times n$ matrix such that its entry in the i -th row and the j -th column is equal to the product of the i -th row of A and the j -th column of B :

$$C = C_{m \times n} = (c_{ij}), \quad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}, \\ i = 1, 2, \dots, m; \quad j = 1, 2, \dots, n.$$

In general, the product of matrices is not commutative: $AB \neq BA$.

Two matrices A and B are said to **commute**, if $AB = BA$. For example, the unit matrix commutes with all matrices; diagonal matrices commute.

Given an $m \times n$ matrix $A = (a_{ij})$, the **transpose of A** is the $n \times m$ matrix $A^T = (a_{ij}^T)$ obtained from A by interchanging its rows and columns. This means that the rows of the matrix A are the columns of the matrix A^T :

$$a_{ij}^T = a_{ji}.$$

A square matrix is called a **symmetric matrix**, if A is equal to the transpose of A :

$$A = A^T.$$

A square matrix is called a **skew-symmetric** (or **antisymmetric** or **antimetric**) **matrix**, if A is equal to the opposite of its transpose:

$$A = -A^T.$$

1.2. Determinants

Let f be a function defined on the set of all square matrices $A = (a_{ij})$ and associates each square matrix with a unique number (real or complex), then $f(A)$ is called the determinant of A . It is also denoted by

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \quad (1)$$

or $|A|$ or Δ .

The determinant of the second order corresponding to the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is calculated according to the formula:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

The determinant of the third order corresponding to the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is calculated by the formula:

$$\begin{aligned} \Delta &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \\ &= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11}. \end{aligned}$$

Symbols a_{ij} are called **elements of determinant**.

The determinant of the order $n-1$, obtained by crossing the row and the column on which intersection the element a_{ij} is located, is called the **minor M_{ij} of the element a_{ij}** .

The **algebraic cofactor of the element a_{ij}** is defined as the minor M_{ij} with the sign $(-1)^{i+j}$. It is denoted by the symbol A_{ij} :

$$A_{ij} = (-1)^{i+j}M_{ij}.$$

Properties of Determinants

1. The value of a determinant remains unchanged if both rows and columns are interchanged.

2. If any two rows (or columns) of a determinant are interchanged, then the sign of the determinant changes.
3. If a matrix has a zero-row or zero-column, then the determinant is equal to zero.
4. If a determinant has two equal rows (or columns) then it is equal to zero.
5. If two rows (or columns) of a determinant are proportional to each other then the determinant is equal to zero.
6. If all the elements of a row (or column) of a determinant is multiplied by a non-zero constant, then the determinant gets multiplied by a similar constant.
7. If each element of i-th row (column) of a determinant is the sum of two summands then such a determinant is equal to the sum of two determinants, of which one row (column) consists of the first addend, and in the second one - from the other; another elements of all three determinants are the same.
8. A determinant holds its value, if a row (column) is multiplied by a number and then is added to another one.
9. The determinant is equal to the sum of products of elements of any row (column) on their algebraic cofactors:

$$\Delta = \sum_k a_{ik} A_{ik} = \sum_k a_{ki} A_{ki}.$$

These formulas are called the **expanding a determinant by the i-th row and the i-th column, respectively**.

10. The determinant of a triangular matrix is equal to the product of the elements on the principle diagonal.
11. The sum of products of elements of any row (column) with the corresponding algebraic cofactors of elements of any other row (column) is zero.

1.3. Inverse Matrix

If the determinant of a square matrix is equal to zero, then the matrix is called **singular**; otherwise, if $\det A \neq 0$, the matrix A is called **regular**.

A matrix A^{-1} is called an **inverse** of the matrix A if

$$AA^{-1} = A^{-1}A = E,$$

where E is an identity matrix.

An inverse matrix is defined only for regular matrix.

If each element of a square matrix A is replaced by its cofactor, then the transpose of the matrix obtained is called the **adjoint** of the matrix A :

$$\text{adj}A = A^* = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

An adjoint matrix is also called an **adjugate matrix**.

An inverse matrix is calculated by the formula

$$A^{-1} = \frac{1}{\det A} \cdot A^*.$$

Such method for finding the inverse matrix is called the **adjoint matrix method**.

We can also calculate inverse matrices by the **method of elementary transformations**.

The following operations is called **elementary transformations** for finding the inverse matrix:

- 1) interchanging the rows of the matrix;
- 2) multiplying a row by a nonzero number.
- 3) adding a row to another row.

Let A be a regular matrix. The inverse matrix A^{-1} can be found using the elementary transformations of the following extended matrix

$$(A|E) = \left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 & 0 & \dots & 1 \end{array} \right),$$

where E is the identity matrix of the corresponding order.

Using elementary row operations we transform the extended matrix to the form $(E|B)$. Then $B = A^{-1}$.

1.4. Matrix Rank

A determinant of order k is called the **minor of order k** of the matrix $A = A_{m \times n}$, if it is obtained by deleting $m - k$ rows and the $n - k$ columns of matrix A .

The **rank $r(A)$** of the matrix A is the maximal order of its non-zero minors.

Bypass Minor Method

If in a matrix A there exists nonzero minor M of order r and all minors bordering it (that is, minors of order $r + 1$ and containing M) are equal to zero then $rank A = r$.

To find the rank of a matrix A , we can assume that the non-zero minor is in the upper left corner of A .

Elementary Transformation Method

The rank of a matrix can be evaluated by applying of the operations:

1. Interchanging two rows or columns.
2. Multiplying a row (column) by a nonzero number.
3. Multiplying a row (column) by a number and adding the result to another row (column).

variables to the right-hand part of equations. Then using the method of back substitution we represent the **main** unknowns x_r, x_{r-1}, \dots, x_1 as a linear combinations of $x_{r+1}, x_{r+2}, \dots, x_n$. Obtained solution is called **the general solution** of the system.

If, after the reducing a system to a row echelon form, it contains an equation of form $0 = d$, where $d \neq 0$, then the system is inconsistent.

2. TASKS

I. Solve the problems.

1. Let $A = \begin{pmatrix} 5 & 8 & 4 \\ 3 & 2 & 5 \\ 7 & 6 & 0 \end{pmatrix}$. Find a matrix B , such as $A + B = E$.

2. Find the sum of matrices $A^2 + A + E$, if $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

3. Find a value of the matrix polynomial $2A^2 + 3A + 5E$, if $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 4 & 1 & 1 \end{pmatrix}$.

4. Solve the matrix equation $A + X = B \cdot C$, where

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 3 & -4 \\ 5 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

5. Let $A = \begin{pmatrix} 7 & 0 & 2 & -1 \\ 4 & 2 & 3 & 6 \\ -7 & 1 & -4 & 3 \\ 8 & 3 & 6 & 1 \end{pmatrix}$. Find a matrix B , such as $A + B = E$.

6. Find a value of the matrix polynomial $\varphi(A) = A^2 - E$, if

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 0 & -3 & 1 \end{pmatrix}.$$

7. Solve the matrix equation $X - A \cdot B = C$, where

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 2 \\ 0 & 2 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 4 & 1 & 2 \\ 0 & -1 & 3 \\ 6 & 1 & 0 \end{pmatrix}.$$

8. Solve the matrix equation $A^2 + X = B^2$, where

9. Solve the matrix equation $X - A^2 \cdot B = 0$, where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -3 & 2 & 2 \\ 1 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 & 3 \\ -2 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix}.$$

10. Find a value of the matrix polynomial $\varphi(A, B) = A \cdot B + B^2$, if

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 2 \\ 1 & -1 & 1 \\ 3 & 0 & 2 \end{pmatrix}$$

11. Find such unknown matrix elements x and y as $AB = BA$, where

$$A = \begin{pmatrix} 1 & x \\ y & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

12. Find a value of the matrix polynomial $\varphi(A, B) = A^2 + E - B$, if

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -2 & 3 & 1 \\ 0 & 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

13. Solve the matrix equation $A \cdot B - X = C$, where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 3 & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 3 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 1 & 3 & 0 \end{pmatrix}.$$

14. Find such unknown matrix elements x and y as $AB = BA$, where

$$A = \begin{pmatrix} x & y \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

15. Find a value of the matrix polynomial $\varphi(A, B) = 2A + B^2 - E$, if

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \\ -2 & 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

16. Find the sum of matrices $2A^2 + 3A$, if $A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & 1 \\ 4 & 0 & 1 \end{pmatrix}$.

17. Solve the matrix equation $A + 2X = B^2$, where

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}.$$

18. Find a value of the matrix polynomial $\varphi(A) = 6E - 4A^2$, if

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

19. Find such unknown matrix elements as

$$\begin{pmatrix} x & 0 & 2 \\ 2 & 1 & y \\ 1 & -2 & 0 \end{pmatrix} + \begin{pmatrix} 4 & y & -2 \\ 1 & 6 & 4 \\ z & 6 & x \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

20. Solve the matrix equation $5A + 2X = B^2$, where

$$A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & 3 & 1 \\ 0 & -2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ -2 & 0 & 1 \end{pmatrix}.$$

21. Find a value of the matrix polynomial $\varphi(A, B) = 2A \cdot B - A^2$, if

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -1 & 1 \\ 0 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ -1 & 2 & 3 \end{pmatrix}.$$

22. Solve the matrix equation $X + A \cdot B = 2B$, where

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 2 \end{pmatrix}.$$

23. Find the sum of matrices $E + 2A + A^3$, if $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix}$.

24. Find a value of the matrix polynomial $\varphi(A) = 2(A - E) \cdot A$, if

$$A = \begin{pmatrix} 0 & 1 & 2 \\ -2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}.$$

25. Solve the matrix equation $A \cdot B - 2X = 3B$, where

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix}.$$

II. Calculate the determinants.

1.
$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ 1 & 2 & -2 & 0 \\ -1 & 3 & 0 & 2 \\ 2 & 1 & -1 & 3 \end{vmatrix}.$$

2.
$$\begin{vmatrix} 6 & 3 & 0 & 3 \\ 4 & 4 & 2 & 1 \\ 0 & 4 & 4 & 2 \\ 7 & 7 & 8 & 5 \end{vmatrix}.$$

3.
$$\begin{vmatrix} 3 & 7 & -2 & 4 \\ -3 & -2 & 6 & -4 \\ 5 & 5 & -3 & 2 \\ 2 & 6 & -5 & 3 \end{vmatrix}.$$

4.
$$\begin{vmatrix} 3 & 5 & 7 & 2 \\ 1 & 2 & 3 & 4 \\ -2 & -3 & 3 & 2 \\ 1 & 3 & 5 & 4 \end{vmatrix}.$$

5.
$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \end{vmatrix}.$$

6.
$$\begin{vmatrix} 1 & -2 & 3 & 4 \\ 2 & 1 & -4 & 3 \\ 3 & -4 & -1 & -2 \\ 4 & 3 & 2 & -1 \end{vmatrix}.$$

7.
$$\begin{vmatrix} -1 & -1 & -1 & -1 \\ -1 & -2 & -4 & -8 \\ -1 & -3 & -9 & -27 \\ -1 & -4 & -16 & -64 \end{vmatrix}.$$

8.
$$\begin{vmatrix} 8 & 7 & 2 & 0 \\ -8 & 2 & 7 & 10 \\ 4 & 4 & 4 & 5 \\ 0 & 4 & -3 & 2 \end{vmatrix}.$$

9.
$$\begin{vmatrix} 2 & 3 & -3 & 4 \\ 2 & 1 & -1 & 2 \\ 6 & 2 & 1 & 0 \\ 2 & 3 & 0 & -5 \end{vmatrix}.$$

10.
$$\begin{vmatrix} 3 & -1 & 4 & 2 \\ 5 & 2 & 0 & 1 \\ 0 & 2 & 1 & -3 \\ 6 & -2 & 9 & 8 \end{vmatrix}.$$

11.
$$\begin{vmatrix} 1 & -1 & 3 & 0 \\ 2 & 0 & 1 & 3 \\ -1 & 1 & 2 & 1 \\ 4 & 0 & 1 & -2 \end{vmatrix}.$$

12.
$$\begin{vmatrix} 1 & 2 & 4 & 1 \\ 1 & 0 & 4 & 1 \\ 2 & -1 & 0 & 1 \\ -1 & 2 & 1 & 0 \end{vmatrix}.$$

13.
$$\begin{vmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}.$$

14.
$$\begin{vmatrix} 4 & 2 & 1 & -1 \\ 1 & 1 & 2 & 3 \\ -2 & 4 & 2 & 2 \\ 1 & 1 & 3 & 1 \end{vmatrix}.$$

15.
$$\begin{vmatrix} -2 & 0 & 4 & 1 \\ -4 & 2 & 8 & 2 \\ 5 & 1 & 3 & 1 \\ -1 & -1 & 0 & 4 \end{vmatrix}.$$

16.
$$\begin{vmatrix} 1 & 2 & 1 & 3 \\ -2 & -3 & 0 & 1 \\ 3 & 6 & 5 & 2 \\ 2 & 9 & 4 & 1 \end{vmatrix}.$$

17.
$$\begin{vmatrix} 8 & 4 & -3 & 0 \\ 4 & 2 & 3 & 1 \\ 2 & 3 & 2 & -1 \\ 6 & 2 & -2 & 5 \end{vmatrix}.$$

18.
$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 \\ 4 & 3 & -2 & -1 \end{vmatrix}.$$

19.
$$\begin{vmatrix} 1 & -2 & 4 & 3 \\ 5 & 3 & 7 & 1 \\ -2 & 4 & -6 & -6 \\ 7 & -1 & 5 & 7 \end{vmatrix}.$$

20.
$$\begin{vmatrix} 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{vmatrix}.$$

21.
$$\begin{vmatrix} -2 & 0 & 4 & 1 \\ -5 & 2 & 2 & 1 \\ -10 & 4 & 3 & 1 \\ 5 & -2 & 7 & 6 \end{vmatrix}.$$

22.
$$\begin{vmatrix} 0 & 14 & 2 & 0 \\ 3 & 5 & -2 & 4 \\ -1 & -7 & -1 & 0 \\ 8 & 7 & 9 & 1 \end{vmatrix}.$$

23.
$$\begin{vmatrix} 2 & 0 & 3 & 4 \\ -4 & 1 & -6 & -2 \\ 1 & 9 & 3 & 6 \\ 0 & 3 & 1 & 2 \end{vmatrix}.$$

24.
$$\begin{vmatrix} 2 & 1 & -2 & 2 \\ 6 & 2 & 1 & 0 \\ 2 & 3 & -3 & 4 \\ 2 & 3 & 0 & -5 \end{vmatrix}.$$

$$25. \begin{vmatrix} -1 & -2 & 0 & 5 \\ 4 & -2 & 6 & -2 \\ 1 & -1 & 3 & -1 \\ 3 & 7 & 0 & -5 \end{vmatrix}.$$

III. Solve the problems.

1. Solve the inequality: $\left| \frac{2x}{x^2} - \frac{5}{4} \right| \geq 3.$

2. Solve the equation: $\begin{vmatrix} x & 1 & 1 \\ 1 & 1 & x \\ 1 & x & 1 \end{vmatrix} = 0.$

3. Prove the identity: $\left| \begin{matrix} \log_a b & \log_{a^2} b \\ \log_{b^2} a & \log_b a \end{matrix} \right| = \frac{3}{4}.$

4. Solve the inequality: $\left| \begin{matrix} x-3 & -2x \\ 1 & \frac{1}{x+2} \end{matrix} \right| < 0.$

5. Prove the identity: $\left| \begin{matrix} \operatorname{tg} \frac{\pi}{5} & \cos \frac{\pi}{3} \\ 2 \sin \frac{\pi}{6} & \operatorname{ctg} \frac{\pi}{5} \end{matrix} \right| = \frac{1}{2}.$

6. Solve the inequality: $\left| \begin{matrix} \frac{5}{x-3} & \frac{8}{x+3} \\ \frac{x-4}{8} & \frac{1}{5} \end{matrix} \right| \leq -1.$

7. Solve the equation: $\left| \begin{matrix} \cos^2 x & 2 \cos x \\ 1 & 1 \end{matrix} \right| = 0.$

8. Solve the equation: $\left| \begin{matrix} x & x-2 \\ -5 & x-2 \end{matrix} \right| = 0.$

9. Solve the inequality: $\left| \begin{matrix} x-1 & 2 \\ 3 & \frac{1}{x+2} \end{matrix} \right| \geq 1.$

10. Prove the identity: $\begin{vmatrix} \cos \frac{\pi}{12} & \sin \frac{\pi}{6} \\ \cos \frac{\pi}{6} & \cos \frac{\pi}{12} \end{vmatrix} = \frac{1}{2}$.

11. Solve the equation: $\begin{vmatrix} \sin x & \cos 2x \\ 2 & 3 \cos x \end{vmatrix} = 0$.

12. Solve the equation: $\begin{vmatrix} x & x+2 \\ x-2 & 2x-5 \end{vmatrix} = 0$.

13. Solve the equation: $\begin{vmatrix} |x+1| & 2 \\ -1 & -1 \end{vmatrix} = 0$.

14. Solve the equation: $\begin{vmatrix} 3^x & 1 & 1 \\ 1 & 1 & 3^x \\ 1 & 3^x & 1 \end{vmatrix} = 0$.

15. Solve the inequality: $\begin{vmatrix} x^2 - 4 & 2 \\ x + 5 & 1 \end{vmatrix} \leq 1$.

16. Prove the identity: $\begin{vmatrix} 1 & 2 & 3 \\ \log_5 x & -1 & -\log_5 x \\ -\log_5 x & 2 \log_5 x & 1 \end{vmatrix} = 4$.

17. Solve the equation: $\begin{vmatrix} \ln x + 1 & 2 \\ -1 & -1 \end{vmatrix} = 0$.

18. Solve the equation: $\begin{vmatrix} 2^x & 2^x - 2 \\ -5 & 2^x - 2 \end{vmatrix} = 0$.

19. Solve the equation: $\begin{vmatrix} 2^x & 2^x + 2 \\ 2^x - 2 & 2^{x+1} - 5 \end{vmatrix} = 0$.

20. Solve the equation: $\begin{vmatrix} 25^x - 4 & 2 \\ 5^x + 5 & 1 \end{vmatrix} = 1$.

21. Solve the inequality: $\begin{vmatrix} \frac{1}{x-1} & \frac{1}{x+1} \\ \frac{1}{x} & 1 \end{vmatrix} \geq 0$.

22. Solve the equation: $\begin{vmatrix} \cos 2x & 1 \\ 1 & \sin 2x \end{vmatrix} = -\frac{1}{2}$.

23. Solve the inequality: $\begin{vmatrix} \cos x & \sin x \\ -1 & 1 \end{vmatrix} \leq -1$.

24. Solve the equation: $\begin{vmatrix} x & 2x \\ x & x^2 - 1 \end{vmatrix} = -2$.

25. Solve the equation: $\begin{vmatrix} \arcsin x - 1 & \arcsin x + 1 \\ \arcsin x + 1 & \arcsin x - 1 \end{vmatrix} = -4$.

IV. Find an inverse matrix for the matrix A :

a) by method of adjoint matrix;

b) by the elementary transformation method.

1. $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}$.

2. $A = \begin{pmatrix} 3 & 5 & -2 \\ 1 & -3 & 2 \\ 6 & 7 & -3 \end{pmatrix}$.

3. $A = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{pmatrix}$.

4. $A = \begin{pmatrix} 10 & 20 & -30 \\ 0 & 10 & 20 \\ 0 & 0 & 10 \end{pmatrix}$.

5. $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 4 & 1 \\ 1 & 6 & -1 \end{pmatrix}$.

6. $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -3 \\ 6 & 0 & -5 \end{pmatrix}$.

7. $A = \begin{pmatrix} -3 & 1 & -1 \\ -2 & 1 & 1 \\ -3 & 2 & 1 \end{pmatrix}$.

8. $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

9. $A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & -2 \\ 2 & 1 & 1 \end{pmatrix}$.

10. $A = \begin{pmatrix} 1 & -4 & 1 \\ 0 & 2 & -2 \\ 1 & -3 & 1 \end{pmatrix}$.

11. $A = \begin{pmatrix} -5 & -1 & -2 \\ 1 & 2 & -3 \\ 2 & 2 & -2 \end{pmatrix}$.

12. $A = \begin{pmatrix} -1 & 1 & 2 \\ -1 & 2 & -5 \\ 1 & -1 & 4 \end{pmatrix}$.

13. $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 3 & 1 & -1 \end{pmatrix}$.

14. $A = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 5 & 2 \\ 1 & 6 & 3 \end{pmatrix}$.

15. $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 1 & 1 \\ -1 & -2 & 2 \end{pmatrix}$.

16. $A = \begin{pmatrix} 3 & 1 & 2 \\ 5 & 1 & -3 \\ 4 & 1 & -1 \end{pmatrix}$.

17. $A = \begin{pmatrix} -1 & 2 & 3 \\ 1 & -2 & 3 \\ 1 & 2 & -3 \end{pmatrix}$.

18. $A = \begin{pmatrix} -1 & 1 & 2 \\ 2 & 1 & -1 \\ 2 & -1 & 2 \end{pmatrix}$.

19. $A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & -1 \\ -3 & 1 & -1 \end{pmatrix}$.

20. $A = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

21. $A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & 2 \\ -2 & 1 & 1 \end{pmatrix}$.

22. $A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix}.$

23. $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix}.$

24. $A = \begin{pmatrix} -1 & -2 & 3 \\ 0 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}.$

25. $A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix}.$

V. Find the rank of the matrix A :

a) by bypass minor method;

b) by the elementary transformation method.

1. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 2 & 1 \\ -1 & 2 & 7 & 5 \\ -1 & 4 & 12 & 10 \end{pmatrix}.$

2. $\begin{pmatrix} 1 & 2 & 2 & 0 & -1 \\ 3 & -1 & 4 & -2 & 4 \\ 5 & 3 & 10 & 8 & 2 \\ 1 & -5 & 0 & -2 & 6 \end{pmatrix}.$

3. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{pmatrix}.$

4. $\begin{pmatrix} 4 & 3 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 2 & 1 & 1 \end{pmatrix}.$

5. $\begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 2 & 0 & 4 & 0 & 0 \end{pmatrix}.$

6. $\begin{pmatrix} 1 & 2 & 3 & 6 \\ 2 & 3 & 1 & 6 \\ 3 & 1 & 2 & 6 \end{pmatrix}.$

7. $\begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$

8. $\begin{pmatrix} 1 & 2 & 1 & 3 & 4 \\ 3 & 4 & 2 & 6 & 8 \\ 1 & 2 & 1 & 3 & 4 \end{pmatrix}.$

9. $\begin{pmatrix} 1 & 5 & 4 & 3 & 1 \\ 2 & -1 & 2 & -1 & 0 \\ 5 & 3 & 8 & 1 & 1 \end{pmatrix}.$

10. $\begin{pmatrix} 1 & 3 & 4 & -2 & 0 \\ 5 & 1 & 6 & -4 & 3 \\ 4 & -2 & 2 & -2 & 3 \\ 6 & 4 & 10 & -6 & 3 \end{pmatrix}.$

11. $\begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 0 & 4 \\ 1 & 0 & 3 & 6 \\ 1 & -2 & 3 & -2 \end{pmatrix}.$

12. $\begin{pmatrix} -3 & 1 & 0 & 5 & 8 \\ 3 & 0 & -4 & 1 & 0 \\ 0 & 1 & -4 & 6 & 8 \end{pmatrix}.$

13. $\begin{pmatrix} 1 & 1 & 2 & 2 & -2 \\ 2 & 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & -3 & 1 \end{pmatrix}.$

14. $\begin{pmatrix} 1 & 3 & 2 & 0 & 3 \\ 3 & 8 & 5 & -1 & 7 \\ 0 & 4 & 4 & 4 & 8 \\ -1 & -4 & -3 & -1 & -5 \end{pmatrix}.$

15. $\begin{pmatrix} 1 & 2 & 2 & 0 \\ 0 & 3 & 0 & -1 \\ 1 & 0 & 2 & -1 \end{pmatrix}.$

16. $\begin{pmatrix} 0 & -5 & 4 & 0 \\ 3 & 0 & 2 & 3 \\ 0 & 10 & -8 & 1 \end{pmatrix}.$

17. $\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ -1 & 1 & 5 & 3 & 0 \\ -1 & 2 & 5 & 4 & 1 \end{pmatrix}.$

18. $\begin{pmatrix} 5 & 3 & 1 & 7 \\ 4 & 3 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$

19. $\begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & -2 & 4 & 4 \\ -1 & -4 & 7 & 8 \\ 3 & -5 & 13 & 10 \end{pmatrix}.$

20. $\begin{pmatrix} 9 & -8 & 0 & 2 \\ 1 & -3 & 0 & -2 \\ -1 & 0 & 2 & 0 \\ 8 & -5 & 0 & 4 \end{pmatrix}.$

21. $\begin{pmatrix} 1 & 2 & 3 \\ -3 & -3 & -3 \\ 4 & 5 & 6 \\ 0 & -3 & -6 \end{pmatrix}.$

22. $\begin{pmatrix} 1 & 5 & 6 & 6 & 0 \\ -2 & 0 & -2 & 1 & 3 \\ 10 & -6 & 4 & 0 & -4 \end{pmatrix}.$

23. $\begin{pmatrix} 0 & 8 & 4 & 1 \\ 5 & -2 & -1 & -3 \\ 5 & 10 & 3 & -2 \\ 10 & 8 & 2 & -5 \end{pmatrix}.$

24. $\begin{pmatrix} 4 & -3 & 10 & -2 \\ 5 & -4 & 19 & 2 \\ 1 & -1 & 9 & 4 \\ 3 & -2 & 1 & -6 \end{pmatrix}.$

$$25. \begin{pmatrix} -10 & 4 & 6 & 2 \\ -1 & 8 & 7 & -1 \\ -3 & 0 & -3 & -3 \\ 8 & -2 & 6 & 8 \end{pmatrix}.$$

VI. Solve the system of linear algebraic equations:

- a) by Cramer's rule;
b) by matrix method.

$$1. \begin{cases} 2x - y - 3z = 3; \\ 5x + y + 7z = 0; \\ x + 3y + 6z = 1. \end{cases}$$

$$2. \begin{cases} 3x + 5y + 4z = 6; \\ 2x - 3y - z = 2; \\ x + y + z = 2. \end{cases}$$

$$3. \begin{cases} 4x + 9y + 2z = 1; \\ x - y - 5z = -2; \\ 2x + y - 4z = -1. \end{cases}$$

$$4. \begin{cases} x + 2y + 2z = 7; \\ 5x + y - 2z = 8; \\ x - y + 3z = -2. \end{cases}$$

$$5. \begin{cases} x - y + 2z = 1; \\ 2x + 3y - 7z = 6; \\ x + y - 4z = 1. \end{cases}$$

$$6. \begin{cases} 3x - y - 2z = -5; \\ 2x + 2y - 3z = 1; \\ x + 2y - 2z = 2. \end{cases}$$

$$7. \begin{cases} 2x + 3y + 4z = -2; \\ x + 5y - 2z = 3; \\ 3x + 2y + 2z = 1. \end{cases}$$

$$8. \begin{cases} 3x - 4y + z = 1; \\ x + 2y - 2z = 0; \\ 2x - 3y + z = 1. \end{cases}$$

$$9. \begin{cases} 2x - 5y + 4z = 2; \\ x + 2y - z = 1; \\ -x - y + z = 1. \end{cases}$$

$$10. \begin{cases} 2x - 3y + z = 5; \\ x - 5y + 2z = 5; \\ 3x + 3y - z = 5. \end{cases}$$

$$11. \begin{cases} 4x + 5y + 2z = 0; \\ x + y + z = 0; \\ 2x - 3y + 5z = 7. \end{cases}$$

$$12. \begin{cases} 2x + 3y + 2z = 3; \\ x - 2y + 3z = 4; \\ 5x + y + 4z = -2. \end{cases}$$

$$13. \begin{cases} x + 2y + z = -1; \\ 3x - 5y + z = 2; \\ 2x - 3y - 2z = -7. \end{cases}$$

$$14. \begin{cases} 2x + 5y + 3z = 1; \\ x + y - 3z = 5; \\ 3x + 3y - 2z = 1. \end{cases}$$

$$15. \begin{cases} x + 5y + 2z = -1; \\ 2x - y - z = 8; \\ x + 6y + 3z = -3. \end{cases}$$

$$16. \begin{cases} 3x - 5y - 2z = -2; \\ 2x + y - 5z = -2; \\ x + y - 3z = -1. \end{cases}$$

$$17. \begin{cases} x + 3y + 2z = 0; \\ 3x - y - 3z = 2; \\ 3x + 2y + z = 4. \end{cases}$$

$$18. \begin{cases} 2x - 3y - 3z = 1; \\ x + 2y + 4z = 0; \\ 3x - 2y - 2z = 4. \end{cases}$$

$$19. \begin{cases} 2x + 5y + z = 1; \\ x - y + 3z = 2; \\ 3x + 2y + 3z = -1. \end{cases}$$

$$20. \begin{cases} 2x + 3y + 4z = 0; \\ 3x + 2y + 2z = -4; \\ x - 3y + 3z = 1. \end{cases}$$

$$21. \begin{cases} 3x + 4y - 2z = -1; \\ x - 3y + z = 4; \\ 2x + 3y - z = -1. \end{cases}$$

$$22. \begin{cases} 2x - y + z = -2; \\ 3x + 2y + 4z = 0; \\ x + 2y + 2z = 2. \end{cases}$$

$$23. \begin{cases} x + 4y + 2z = 6; \\ 2x + 5y + z = 6; \\ 3x + 2y - 3z = 1. \end{cases}$$

$$24. \begin{cases} 3x + 2y + 2z = 2; \\ 2x + 4y + 3z = -5; \\ 3x + y - 2z = 1. \end{cases}$$

$$25. \begin{cases} 2x + 4y + 7z = 1; \\ x - 3y - z = 1; \\ 6x - 3y + 4z = 1. \end{cases}$$

VII. Solve the system of linear algebraic equations by Gauss' method.

$$1. \begin{cases} x_1 + x_2 + 2x_3 + 3x_4 = 1; \\ 3x_1 - x_2 - x_3 - 2x_4 = -4; \\ 2x_1 + 3x_2 - x_3 - x_4 = -6; \\ x_1 + 2x_2 + 3x_3 - x_4 = -4. \end{cases}$$

$$2. \begin{cases} 2x_1 - 3x_2 + x_3 - x_4 = 6; \\ x_1 + 2x_2 - 4x_3 - 2x_4 = 0; \\ 3x_1 - 3x_2 + x_3 + 2x_4 = 5; \\ 2x_1 + x_2 - 2x_3 + x_4 = 1. \end{cases}$$

$$3. \begin{cases} x_1 + 2x_2 - x_3 + 4x_4 = 3; \\ 4x_1 + 4x_2 + x_3 - 3x_4 = 3; \\ 2x_1 - 2x_2 - 3x_3 + 4x_4 = 0; \\ x_1 + 3x_2 + x_3 + 2x_4 = 5. \end{cases}$$

$$4. \begin{cases} x_1 + x_2 + 2x_3 - 4x_4 = 1; \\ 3x_1 - x_2 - x_3 + x_4 = 4; \\ x_1 - 2x_2 + x_3 + 2x_4 = -2; \\ 2x_1 + 2x_2 + 2x_3 - 6x_4 = 4. \end{cases}$$

$$5. \begin{cases} x_1 - x_2 + 3x_3 - x_4 = 2; \\ 2x_1 - 3x_2 + 5x_3 - 4x_4 = -1; \\ x_1 + x_2 - 5x_3 + 4x_4 = 1; \\ x_1 - 2x_2 + x_3 - 2x_4 = -5. \end{cases}$$

$$6. \begin{cases} 2x_1 - 3x_2 + 4x_3 + 2x_4 = 4; \\ x_1 - 2x_2 + x_3 - 3x_4 = 2; \\ x_1 + 4x_3 + 5x_4 = 3; \\ 3x_1 - 5x_2 + 6x_3 + x_4 = 5. \end{cases}$$

$$7. \begin{cases} x_1 - 2x_2 - 3x_3 - 4x_4 = 0; \\ x_1 + x_2 - 3x_3 - 5x_4 = 2; \\ 2x_1 - 5x_2 + x_3 - 4x_4 = 10; \\ x_1 + x_2 - x_3 + x_4 = 10. \end{cases}$$

$$8. \begin{cases} 2x_1 - 3x_2 + 2x_3 - 5x_4 = -1; \\ x_1 - 2x_2 - 4x_3 + x_4 = -2; \\ 2x_1 + x_2 - 3x_3 + 2x_4 = 11; \\ 3x_1 - 2x_2 - x_3 - x_4 = 6. \end{cases}$$

$$9. \begin{cases} 2x_1 + 4x_2 + 3x_3 - 2x_4 = 0; \\ x_1 + 3x_2 + 4x_3 - x_4 = 1; \\ 3x_1 + 5x_2 + x_3 + x_4 = -5; \\ x_1 - x_2 - 2x_3 + 2x_4 = -6. \end{cases}$$

$$10. \begin{cases} 2x_1 + x_2 - 2x_3 + 4x_4 = 7; \\ 3x_1 - 2x_2 + 5x_3 + x_4 = 1; \\ 2x_1 - 2x_2 + 4x_3 + x_4 = 1; \\ 2x_1 + 3x_2 + 2x_3 + 3x_4 = 0. \end{cases}$$

$$11. \begin{cases} 2x_1 + 5x_2 - x_3 + 4x_4 = 1; \\ x_1 - 6x_2 + 2x_3 - 7x_4 = 3; \\ 3x_1 - 2x_2 - 2x_3 + 6x_4 = 1; \\ 2x_1 + x_2 + x_3 - 2x_4 = 3. \end{cases}$$

$$12. \begin{cases} 3x_1 - x_2 + 5x_3 + 2x_4 = -3; \\ 2x_1 + 2x_2 - 3x_3 + x_4 = 1; \\ x_1 - 4x_2 + 3x_3 - 2x_4 = -2; \\ 3x_1 - 2x_2 + x_3 + x_4 = -3. \end{cases}$$

$$13. \begin{cases} 4x_1 - 3x_2 - 4x_3 + 5x_4 = 1; \\ 2x_1 + x_2 - 3x_3 - 2x_4 = 2; \\ 2x_1 - 2x_2 + x_3 - 7x_4 = 3; \\ 4x_1 - x_2 - 3x_3 + 2x_4 = 4. \end{cases}$$

$$14. \begin{cases} 2x_1 + 3x_2 - 2x_3 - x_4 = 0; \\ x_1 - 5x_2 + 3x_3 - 2x_4 = 1; \\ 3x_1 + 2x_2 + x_3 + x_4 = 1; \\ x_1 - x_2 - 4x_3 + x_4 = 7. \end{cases}$$

$$15. \begin{cases} x_1 + x_2 - 3x_3 + 4x_4 = -6; \\ 2x_1 + 4x_2 - x_3 + 6x_4 = 1; \\ x_1 + 2x_2 - 2x_3 + 2x_4 = 0; \\ 3x_1 - x_2 + 3x_3 - 4x_4 = -2. \end{cases}$$

$$16. \begin{cases} x_1 + 2x_2 + 3x_3 + x_4 = 1; \\ x_1 - x_2 + 2x_3 + 4x_4 = -2; \\ x_1 + x_2 - 2x_3 + 4x_4 = 2; \\ x_1 + x_2 + 3x_3 + 2x_4 = 0. \end{cases}$$

$$17. \begin{cases} x_1 - 2x_2 + 4x_3 + 2x_4 = 2; \\ 2x_1 - 3x_2 - x_3 - 4x_4 = 6; \\ x_1 + x_2 + 4x_3 - 2x_4 = 1; \\ x_1 - 4x_2 + x_3 + x_4 = 2. \end{cases}$$

$$18. \begin{cases} 2x_1 - x_2 + 2x_3 - 2x_4 = -1; \\ 3x_1 + x_2 + 4x_3 + x_4 = 1; \\ 3x_1 - x_2 - 2x_3 + x_4 = 1; \\ x_1 - 2x_2 - 3x_3 - x_4 = -1. \end{cases}$$

$$19. \begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1; \\ 2x_1 - x_2 + 4x_3 - 2x_4 = 2; \\ x_1 - x_2 + 2x_3 + x_4 = 0; \\ x_1 + x_2 + 2x_3 - 2x_4 = -1. \end{cases}$$

$$20. \begin{cases} 3x_1 - 2x_2 + 2x_3 + x_4 = 2; \\ 2x_1 - 2x_2 + x_3 + 2x_4 = 1; \\ 2x_1 + x_2 + 4x_3 - 3x_4 = 2; \\ 3x_1 - x_2 + 4x_3 - x_4 = 1. \end{cases}$$

$$21. \begin{cases} x_1 + x_2 - 2x_3 + 3x_4 = 2; \\ 2x_1 + 3x_2 + x_3 + 2x_4 = 2; \\ 2x_1 + x_2 - 2x_3 - x_4 = 2; \\ 4x_1 + 3x_2 - 4x_3 - x_4 = 2. \end{cases}$$

$$22. \begin{cases} 2x_1 - 3x_2 + 4x_3 - 2x_4 = -2; \\ x_1 + 2x_2 - 2x_3 + 2x_4 = 2; \\ x_1 - x_2 + 2x_3 + x_4 = -1; \\ x_1 - 2x_2 + 3x_3 - 2x_4 = 0. \end{cases}$$

$$23. \begin{cases} 2x_1 - x_2 + 3x_3 - 4x_4 = 4; \\ 2x_1 + x_2 - 4x_3 + 2x_4 = 0; \\ x_1 - 2x_2 + 3x_3 - 4x_4 = 0; \\ 3x_1 - 4x_2 + 2x_3 - 2x_4 = 0. \end{cases}$$

$$24. \begin{cases} x_1 - 2x_2 + 3x_3 + 2x_4 = -2; \\ 2x_1 + x_2 + x_3 + 2x_4 = -1; \\ 2x_1 - 2x_2 + x_3 + 4x_4 = -2; \\ x_1 + 3x_2 + 2x_3 + x_4 = 2. \end{cases}$$

$$25. \begin{cases} 3x_1 + 2x_2 + x_3 + 4x_4 = 6; \\ 3x_1 - 2x_2 + x_3 - 4x_4 = -2; \\ 2x_1 + 2x_2 - 3x_3 - 4x_4 = 1; \\ 2x_1 - 3x_2 + x_3 + x_4 = 2. \end{cases}$$

VIII. Solve the system of linear homogeneous equations.

$$1. \begin{cases} x_1 - 2x_2 + x_3 - 3x_4 = 0; \\ -x_1 + 3x_2 - 2x_3 + x_4 = 0; \\ 2x_1 + 3x_2 - x_3 + x_4 = 0. \end{cases}$$

$$2. \begin{cases} 2x_1 + 3x_2 - x_3 = 0; \\ 4x_1 - x_2 + 2x_3 = 0; \\ 6x_1 + 2x_2 - x_3 = 0. \end{cases}$$

$$3. \begin{cases} x_1 - 2x_2 + x_3 + x_4 - x_5 = 0; \\ 2x_1 + x_2 - x_3 - x_4 + x_5 = 0; \\ x_1 + 7x_2 - 5x_3 - 5x_4 + 5x_5 = 0; \\ 3x_1 - x_2 - 2x_3 + x_4 - x_5 = 0. \end{cases}$$

$$4. \begin{cases} x_1 + 2x_2 + 3x_3 = 0; \\ 2x_1 + x_2 + x_3 = 0; \\ x_1 - x_2 + 2x_3 = 0. \end{cases}$$

$$5. \begin{cases} 2x_1 + 2x_2 - 3x_3 - x_4 = 0; \\ x_1 - x_2 + 2x_3 - x_4 = 0; \\ 3x_1 + x_2 - x_3 - 2x_4 = 0; \\ x_1 - 2x_2 + 3x_3 - x_4 = 0. \end{cases}$$

$$6. \begin{cases} 3x_1 + 2x_2 - x_3 = 0; \\ 2x_1 - x_2 + 3x_3 = 0; \\ x_1 + x_2 - x_3 = 0. \end{cases}$$

$$7. \begin{cases} x_1 - 3x_3 + 4x_4 = 0; \\ x_1 + x_2 - 2x_3 + x_4 = 0; \\ x_1 + 2x_2 - 5x_3 + 5x_4 = 0. \end{cases}$$

$$8. \begin{cases} 3x_1 + 2x_2 + x_3 = 0; \\ 2x_1 + 5x_2 + 3x_3 = 0; \\ 3x_1 + 4x_2 + 2x_3 = 0. \end{cases}$$

$$9. \begin{cases} -2x_1 + 2x_2 + 3x_3 + x_4 = 0; \\ x_2 - x_3 + x_4 = 0; \\ 3x_1 + 2x_2 + x_3 - 2x_4 = 0; \\ 2x_1 + x_2 + x_3 - x_4 = 0. \end{cases}$$

$$10. \begin{cases} 3x_1 - x_2 + 2x_3 = 0; \\ 2x_1 + 3x_2 - 5x_3 = 0; \\ x_1 + x_2 + x_3 = 0. \end{cases}$$

$$11. \begin{cases} x_1 + x_2 + x_3 = 0; \\ 2x_2 + x_3 + 3x_4 = 0; \\ -4x_2 - 2x_3 - 6x_4 = 0; \\ x_1 + 2x_2 + x_3 - x_4 = 0. \end{cases}$$

$$12. \begin{cases} x_1 + 2x_2 = 0; \\ -x_1 + x_2 + 2x_3 = 0; \\ 3x_1 + x_3 = 0. \end{cases}$$

$$13. \begin{cases} x_1 + x_2 - x_4 = 0; \\ x_1 + 2x_2 - 2x_3 - x_4 = 0; \\ -2x_1 - 2x_2 + 2x_4 = 0; \\ 2x_2 + x_3 - x_4 = 0. \end{cases}$$

$$14. \begin{cases} 3x_1 + 2x_2 + 2x_3 = 0; \\ x_1 + 3x_2 + x_3 = 0; \\ 5x_1 + 3x_2 + 4x_3 = 0. \end{cases}$$

$$15. \begin{cases} x_1 + 2x_2 - 2x_3 - 2x_4 = 0; \\ 2x_1 + 2x_2 + x_3 + x_4 = 0; \\ x_1 + 2x_2 + 3x_3 - 3x_4 = 0; \\ 3x_1 + 3x_2 + 2x_3 - x_4 = 0. \end{cases}$$

$$16. \begin{cases} 3x_1 + 5x_2 - 2x_3 = 0; \\ x_1 - 3x_2 + 2x_3 = 0; \\ 6x_1 + 7x_2 - 3x_3 = 0. \end{cases}$$

$$17. \begin{cases} x_1 - 2x_2 + x_4 = 0; \\ -x_2 + x_3 + x_4 = 0; \\ 3x_1 + 2x_2 + x_3 - 2x_4 = 0; \\ x_1 + x_2 - x_4 = 0. \end{cases}$$

$$18. \begin{cases} 3x_1 + 2x_2 + x_3 = 0; \\ 2x_1 + 3x_2 + x_3 = 0; \\ 2x_1 + x_2 + 3x_3 = 0. \end{cases}$$

$$19. \begin{cases} 2x_1 + 3x_2 - x_3 + 3x_4 = 0; \\ 4x_1 + 6x_2 + 2x_3 + x_4 = 0; \\ 2x_1 + 3x_2 + 3x_3 - 2x_4 = 0; \\ 6x_1 + 3x_2 - 2x_3 + 2x_4 = 0. \end{cases}$$

$$20. \begin{cases} x_1 + x_2 + 2x_3 = 0; \\ 2x_1 - x_2 + 2x_3 = 0; \\ 4x_1 + x_2 + 4x_3 = 0. \end{cases}$$

$$21. \begin{cases} 3x_1 - 2x_2 + 3x_3 + x_4 = 0; \\ x_1 + x_2 - x_3 + 3x_4 = 0; \\ 4x_1 - x_2 + 2x_3 + 4x_4 = 0; \\ 2x_1 - 3x_2 + 4x_3 - 2x_4 = 0. \end{cases}$$

$$22. \begin{cases} 2x_1 + 3x_2 + 2x_3 = 0; \\ x_1 + 2x_2 - 3x_3 = 0; \\ 3x_1 + 4x_2 + x_3 = 0. \end{cases}$$

$$23. \begin{cases} x_1 - 3x_2 + x_3 + 2x_4 = 0; \\ 2x_1 + x_2 - 3x_3 + x_4 = 0; \\ 3x_1 - 2x_2 - 2x_3 + 3x_4 = 0; \\ x_1 + 4x_2 - 4x_3 - x_4 = 0. \end{cases}$$

$$24. \begin{cases} x_1 - x_2 + x_3 = 0; \\ 2x_1 + x_2 + x_3 = 0; \\ 2x_2 - x_3 = 0. \end{cases}$$

$$25. \begin{cases} x_1 + x_2 - 2x_3 + 3x_4 = 0; \\ 2x_1 + 2x_2 - x_3 + 2x_4 = 0; \\ 3x_1 + 3x_2 - 3x_3 + 5x_4 = 0; \\ x_1 + x_2 + x_3 - x_4 = 0. \end{cases}$$

IX. Investigate the system for consistency and find its solutions.

$$1. \begin{cases} x_1 + 5x_2 + 4x_3 = 1; \\ 2x_1 + 10x_2 + 8x_3 = 3; \\ 3x_1 + 15x_2 + 12x_3 = 5. \end{cases}$$

$$2. \begin{cases} x_1 - 3x_2 + 2x_3 = -1; \\ x_1 + 9x_2 + 6x_3 = 3; \\ x_1 + 3x_2 + 4x_3 = 1. \end{cases}$$

$$3. \begin{cases} x_1 + 3x_2 + 5x_3 + 7x_4 + 9x_5 = 1; \\ x_1 - 2x_2 + 3x_3 - 4x_4 + 5x_5 = 2; \\ 2x_1 + 11x_2 + 12x_3 + 25x_4 + 22x_5 = 4. \end{cases}$$

$$4. \begin{cases} 2x_1 - 2x_2 + 3x_3 + x_4 = 0; \\ 4x_1 - 4x_2 + 7x_3 - x_4 = 4/3; \\ x_1 - x_2 + 3x_3 - 2x_4 = 0; \\ -x_1 + 2x_2 - x_3 + 3x_4 = 0. \end{cases}$$

$$5. \begin{cases} 3x_1 - 5x_2 + 2x_3 + 4x_4 = 2; \\ 7x_1 - 4x_2 + x_3 + 3x_4 = 5; \\ 5x_1 + 7x_2 - 4x_3 - 6x_4 = 3. \end{cases}$$

$$6. \begin{cases} 4x_1 + 2x_2 + x_3 - x_4 = 0; \\ 2x_1 + x_2 + x_3 - x_4 = 0; \\ x_1 - x_2 + 2x_3 - x_4 = 0; \\ x_1 + 3x_2 - 2x_3 + x_4 = 1. \end{cases}$$

$$7. \begin{cases} x_1 + 2x_2 + 3x_4 + x_5 = 1; \\ -x_1 + 2x_2 + 4x_3 + x_4 + 2x_5 = 0; \\ 4x_2 + 4x_3 + 4x_4 + 3x_5 = 0; \\ -2x_1 + 4x_3 - 2x_4 + x_5 = 3. \end{cases}$$

$$8. \begin{cases} 2x_1 + 5x_2 - 3x_3 = 2; \\ 3x_1 - 6x_2 - x_3 = 10; \\ 2x_1 - x_2 - 2x_3 = 4; \\ x_1 + x_2 + 2x_3 = 8. \end{cases}$$

$$9. \begin{cases} x_2 + 3x_3 - 2x_4 + x_5 = 1; \\ 2x_1 - x_2 + x_3 - x_4 + 3x_5 = 0; \\ 2x_1 + 4x_3 - 3x_4 + 4x_5 = 2; \\ -2x_2 - 2x_3 + x_4 + 2x_5 = 4. \end{cases}$$

$$10. \begin{cases} x_1 + 2x_2 - x_3 + x_4 = 7; \\ 2x_1 - x_2 + x_3 - x_4 = -2; \\ 3x_1 + 3x_2 - 2x_3 + x_4 = 10; \\ x_1 - 3x_2 + 2x_3 - 2x_4 = 3. \end{cases}$$

$$11. \begin{cases} 2x_1 + 2x_3 - x_4 + 2x_5 = 1; \\ x_1 + 3x_2 + x_3 + x_5 = 0; \\ 3x_1 + 3x_2 + 3x_3 - x_4 + 3x_5 = 2; \\ x_1 - 3x_2 + x_3 - x_4 + x_5 = -3. \end{cases}$$

$$12. \begin{cases} 2x_1 - x_2 + 3x_3 - x_4 = -2; \\ 3x_1 + 2x_2 - x_3 + x_4 = 3; \\ x_1 - 2x_2 + 3x_3 + 2x_4 = 0; \\ 2x_1 + 2x_2 - x_3 + 2x_4 = 4. \end{cases}$$

$$13. \begin{cases} x_1 + 3x_3 + 2x_4 - x_5 = 2; \\ 2x_1 + x_2 + 4x_3 - 2x_4 = 1; \\ 3x_1 + x_2 + 7x_3 - x_5 = -1; \\ -x_1 - x_2 - x_3 + 4x_4 - x_5 = 3. \end{cases}$$

$$14. \begin{cases} 2x_1 - x_2 + 3x_3 + x_4 = 0; \\ x_1 + 2x_2 - x_3 - x_4 = 5; \\ 3x_1 + x_2 + 2x_3 = 5; \\ x_1 - 3x_2 + 4x_3 + 2x_4 = -5. \end{cases}$$

$$15. \begin{cases} x_1 + x_2 + 3x_3 - x_4 + 2x_5 = 0; \\ 2x_1 + 2x_4 - x_5 = 1; \\ 3x_1 + x_2 + 3x_3 + x_4 + x_5 = 2; \\ -x_1 + x_2 + 3x_3 - 3x_4 + 3x_5 = -2. \end{cases}$$

$$16. \begin{cases} x_1 + 2x_2 - 3x_3 + x_4 = -1; \\ 2x_1 - x_2 + x_3 - 3x_4 = 5; \\ 3x_1 + x_2 - 2x_3 - 2x_4 = 4; \\ x_1 - 2x_2 + 2x_3 + 2x_4 = -1. \end{cases}$$

$$17. \begin{cases} x_1 + 2x_2 + 3x_3 - x_4 + 2x_5 = 1; \\ x_2 + 4x_4 + x_5 = 0; \\ x_1 + 3x_2 + 3x_3 + 3x_4 + 3x_5 = 2; \\ x_1 + x_2 + 3x_3 - 5x_4 + x_5 = 3. \end{cases}$$

$$18. \begin{cases} 3x_1 + 2x_2 + x_3 + 4x_4 = 0; \\ 2x_1 + 3x_2 + x_3 + 2x_4 = 1; \\ x_1 - x_2 + 2x_4 = -1; \\ 5x_1 + 5x_2 + 2x_3 + 6x_4 = 1. \end{cases}$$

$$19. \begin{cases} 2x_1 + x_2 + 4x_3 + x_5 = -1; \\ 2x_2 + 4x_3 + x_4 + 2x_5 = 2; \\ 2x_1 + 3x_2 + 8x_3 + x_4 + 3x_5 = 0; \\ -2x_1 + x_2 + x_4 + x_5 = -3. \end{cases}$$

$$20. \begin{cases} x_1 + x_2 + 2x_3 + 3x_4 = 1; \\ 2x_1 - x_2 + 2x_3 = 1; \\ 3x_1 + 4x_3 + 3x_4 = 2; \\ -x_1 + 2x_2 + 3x_4 = 0. \end{cases}$$

$$21. \begin{cases} x_2 - x_3 + 2x_4 = 1; \\ 3x_1 + x_2 + 2x_3 + x_4 = 1; \\ 3x_1 + 2x_2 + x_3 + 3x_4 = 0; \\ -3x_1 - 3x_3 + x_4 = 2. \end{cases}$$

$$22. \begin{cases} 2x_1 + x_2 + 3x_3 + 4x_4 = 0; \\ x_2 + 2x_3 + x_4 = 1; \\ 2x_1 + 2x_2 + 5x_3 + 5x_4 = 1; \\ 2x_1 + x_3 + 3x_4 = -1. \end{cases}$$

$$23. \begin{cases} 2x_1 + x_3 + 3x_4 = 0; \\ x_1 + 3x_2 + 4x_4 = 1; \\ 3x_1 + 3x_2 + x_3 + 7x_4 = 0; \\ x_1 - 3x_2 + x_3 - x_4 = 1. \end{cases}$$

$$24. \begin{cases} x_1 + x_2 + 3x_3 + 2x_4 = 1; \\ x_1 + 2x_2 + 6x_3 + 4x_4 = 0; \\ 2x_1 + 3x_2 + 9x_3 + 6x_4 = 1; \\ x_2 + 3x_3 + 2x_4 = -1. \end{cases}$$

$$25. \begin{cases} 4x_1 + x_2 + 2x_3 + 3x_4 = 0; \\ x_1 + 2x_2 + 2x_3 + x_4 = 1; \\ 5x_1 + 3x_2 + 4x_3 + 4x_4 = 0; \\ 3x_1 - x_2 - 2x_4 = 3. \end{cases}$$

RECOMMENDED LITERATURE LIST

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